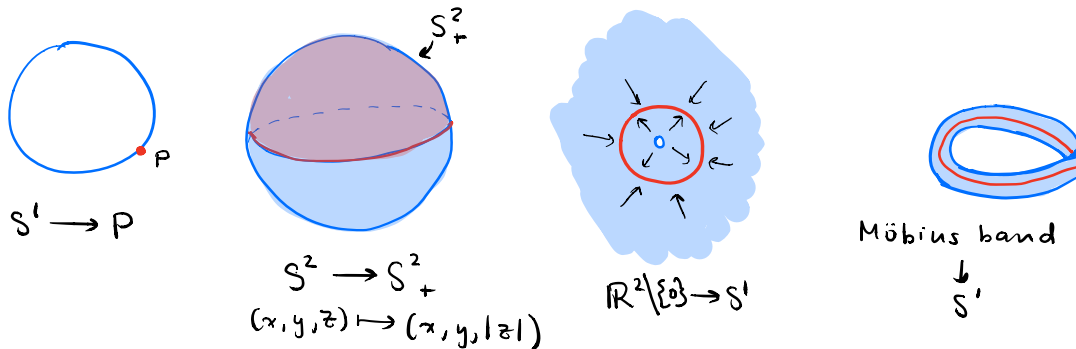


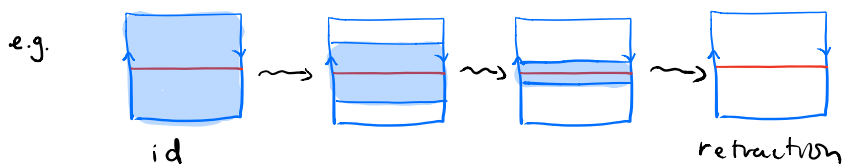
Deformation Retracts

Here are some examples of retractions:



How are the second two different from the first two?

In the second two, we can vary the identity map continuously in the larger space to get to the retraction.



There's no way to do this in the first two examples.

Def: Let $A \subseteq X$ a subspace. A is a deformation retract of X if the identity map id_X is homotopic to a map sending all of X into A s.t. all points of A are fixed during the homotopy.

That is, \exists a continuous $H: X \times I \rightarrow X$ s.t.

- $H(x, 0) = x$,
- $H(a, t) = a \quad \forall a \in A$
- $H(x, 1) \in A$.

Such a homotopy H is called a deformation retraction of X onto A .

Note that the function $r: X \rightarrow A$ defined $r(x) = H(x, 1)$ is a retraction of X onto A , and H is a homotopy between the identity id_X and $j \circ r$, where $j: A \rightarrow X$ is the inclusion.

Ex: If $X = \mathbb{R}^n \setminus \{0\}$, then $H: X \times I \rightarrow X$ defined

$$H(x, t) = \frac{tx}{\|x\|} + (1-t)x \text{ is a deformation retract onto } S^{n-1}$$

(Note: $H(x, t) = \left((1-t) + \frac{t}{\|x\|} \right) x$, and $1-t \geq 0$, $\frac{t}{\|x\|} \geq 0$, and they are never 0 simultaneously.)

$$\text{If } a \in S^{n-1}, \text{ then } H(a, t) = \frac{ta}{1} + (1-t)a = a.$$

How do the fundamental groups of A and X compare if A is a def. retract of X ?

Lemma: $h, k: (X, x_0) \rightarrow (Y, y_0)$ continuous maps. If h and k are homotopic and $x_0 \mapsto y_0$ throughout the homotopy, then h_* and k_* are equal.

(i.f. $k: x_0 \mapsto y_1$, then $k_* = \hat{\alpha} \circ h_*$, α the path from y_0 to y_1)
 \downarrow
 $\pi_1(Y, y_0) \rightarrow \pi_1(Y, y_1)$

Pf: If f is a loop at x_0 in X , then

$$\begin{array}{c} I \times I \xrightarrow{f \times \text{id}} X \times I \xrightarrow{H} Y \\ \text{path} \end{array}$$

is a homotopy between $h \circ f$ and $k \circ f$. \square

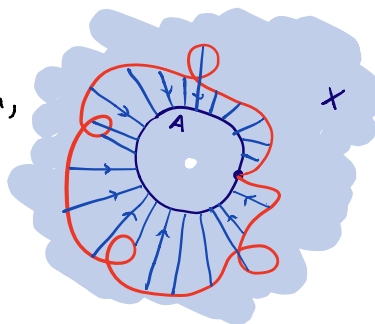
Recall from a previous section that if A is a retract of X , then the inclusion $j: A \hookrightarrow X$ induces an injective homomorphism j_* on fundamental groups. If A is a deformation retract of X , we can say something stronger:

Thm: If A is a deformation retract of X and $j: (A, x_0) \rightarrow (X, x_0)$ the inclusion, then $j_*: \pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$ is an isomorphism of fundamental groups.

Pf: Let $H: X \times I \rightarrow X$ be a deformation retraction of X onto A , so $H(x, 0) = x$, $H(x, 1) \in A$, and $H(a, t) = a \forall a \in A$.

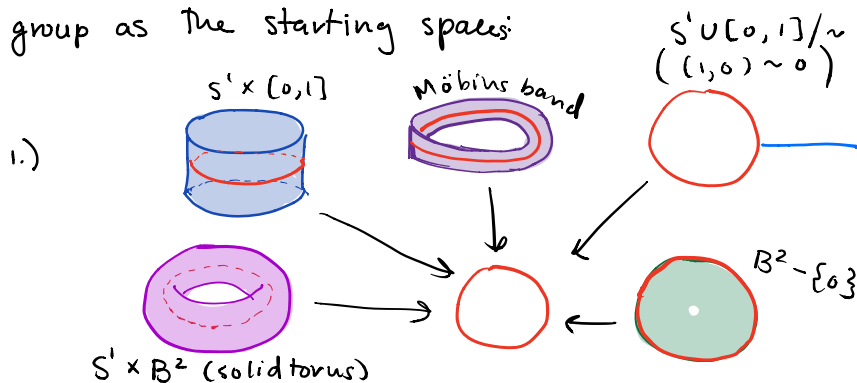
Then if $r: X \rightarrow A$ is the retraction,

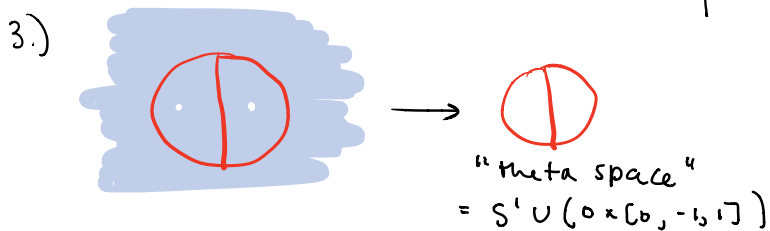
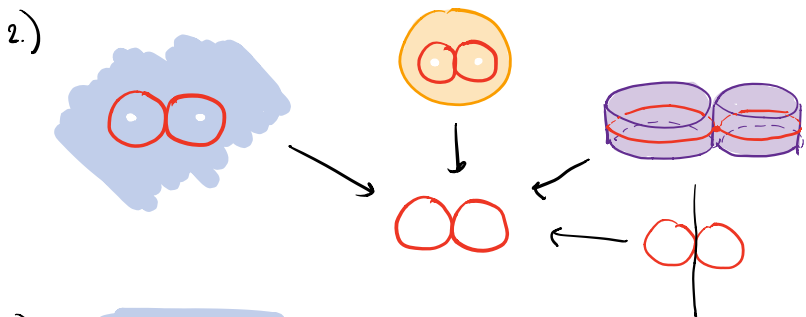
$X \xrightarrow{r} A \xrightarrow{j} X$ is homotopic to the identity, so, $j_* \circ r_* = \text{id}_{\pi_1(X)}$.



$\Rightarrow j_*$ is surjective, so it's an isomorphism. \square

Ex: The following deformation retracts have the same fundamental group as the starting spaces:



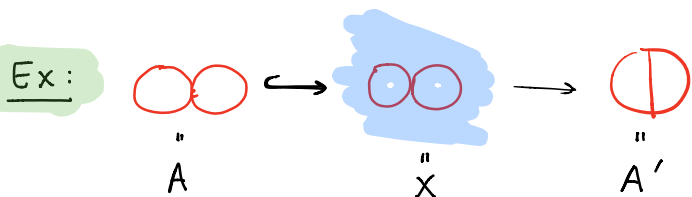


Note that $\mathbb{R}^2 - \{\text{two points}\}$ has a deformation retraction onto both $\bigcirc \bigcirc$ and $\bigcirc \text{---}$ neither of which is a deformation retract of the other.

Def: let $f: X \rightarrow Y$ and $g: Y \rightarrow X$ be continuous maps. If $g \circ f: X \rightarrow X$ is homotopic to id_X and $f \circ g: Y \rightarrow Y$ is homotopic to id_Y , then f and g are homotopy equivalences, and X and Y have the same homotopy type. f and g are homotopy inverses of each other.

Note: If A is a deformation retract of X , then

$X \xrightarrow{r} A \xrightarrow{j} X$ is homotopic to the identity, and $A \xrightarrow{j} X \xrightarrow{r} A$ is the identity. Thus A and X have the same homotopy type.



Then $A \rightarrow X \rightarrow A' \rightarrow X \rightarrow A$ is homotopic to $A \rightarrow X \rightarrow A$, which

is the identity. The other direction is analogous, so A and A' have the same homotopy type.

This is true in general. i.e....

Claim: If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are homotopy equivalences, then $g \circ f$ is a homotopy equivalence. Thus, having the same homotopy type is an equivalence relation.

Ex: X is contractible if id_X is nullhomotopic.

(e.g. I, B^n, \mathbb{R}^n , etc.)

If $H: X \times I \rightarrow X$ is a homotopy from the identity to a constant map (w/ image $P \in X$, say), then

$P \hookrightarrow X \rightarrow P$ is the identity and $X \xrightarrow{\quad} P \hookrightarrow X$

is homotopic to the identity, via H . so $P \hookrightarrow X$ is a homotopy equivalence.

(Note: H may not be a deformation retraction—i.e. if $H(p,t) \neq p$ for some t)

Thm: If $f: (X, x_0) \rightarrow (Y, y_0)$ is a homotopy equivalence, then $f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ is an isomorphism.

Pf: Let $g: Y \rightarrow X$ be a homotopy inverse for f , where $y_0 \mapsto x_1$.

Set $f(x_1) = y_1$.

Then we have $\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(X, x_1) \xrightarrow{f'_*} \pi_1(Y, y_1)$

$g \circ f$ is homotopic to the identity, so there's a path α from x_0 to x_1 s.t.

$$g_* \circ f_* = \hat{\alpha} \circ (id_x)_* = \hat{\alpha}, \text{ an isomorphism.}$$

Similarly, $f'_* \circ g_*$ is an isomorphism, so g_* is injective and surjective, so it's an isom.

Thus, $f_* = (g_*)^{-1} \circ \hat{\alpha}$ is an isomorphism. \square